SYDNEY GRAMMAR SCHOOL

2013 Assessment Examination

FORM VI
MATHEMATICS EXTENSION 1

Monday 20th May 2013

General Instructions

- Writing time — 1 hour 30 minutes
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 55 Marks

- All questions may be attempted.

Section I – 7 Marks

- Questions 1–7 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 48 Marks

- Questions 8–11 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eight.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 113 boys

Examiner

BR/DNW

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

When $2x^3 + x^2 + kx - 4$ is divided by $(x - 1)$ the remainder is 2. The value of k is: **1**

(A) -7

(B) 7

(C) 1

(D) 3

QUESTION TWO

The velocity v of a certain object is related to the displacement x by $v^2 = (x - 3)^2$. Its acceleration is: **1**

(A) $x - 3$

(B) $2(x - 3)$

(C) $\frac{1}{6}(x - 3)^3$

(D) $\frac{1}{3}(x - 3)^3$

QUESTION THREE

A particle starts at rest 3 m to the right of the origin and moves in simple harmonic motion along the x-axis with period 2 s. The equation of motion is: **1**

(A) $x = 3\sin 2t$

(B) $x = 3\cos 2t$

(C) $x = 3\sin \pi t$

(D) $x = 3\cos \pi t$

QUESTION FOUR

The equation $x^3 - 2x^2 - x + 1 = 0$ has roots α, β and γ. Which of the following is true? **1**

(A) $\alpha + \beta + \gamma = -2$ and $\alpha\beta\gamma = -1$

(B) $\alpha + \beta + \gamma = -2$ and $\alpha\beta\gamma = 1$

(C) $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = -1$

(D) $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = 1$

Exam continues next page ...

QUESTION FIVE

The exact value of $\tan\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$ is: **1**

(A) $\frac{\sqrt{5}}{2}$

(B) $-\frac{\sqrt{5}}{2}$

(C) $\frac{2}{\sqrt{5}}$

(D) $-\frac{2}{\sqrt{5}}$

QUESTION SIX

The double angle formulae can be used to show that $x = 3 - 4\sin^2 t$ is an example of simple harmonic motion. The centre of motion is: **1**

(A) $x = 1$

(B) $x = 2$

(C) $x = 3$

(D) $x = 5$

QUESTION SEVEN

A steel ball is fired at an initial speed of 12 m/s at an angle of elevation of 30° under the influence of gravity alone. The speed of the ball at its maximum height is: **1**

(A) 0 m/s

(B) 6 m/s

(C) $6\sqrt{3}$ m/s

(D) 12 m/s

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) Use the factor theorem to show that $(x+1)$ is a factor of $P(x) = 6x^3 + 5x^2 - 2x - 1$. **1**

(b) Let $f(x) = 4\sin^{-1} 2x$. What is the domain of $f(x)$? **1**

(c) Differentiate $y = \tan^{-1}(3x)$. **2**

(d) The equation $x^3 + 2x^2 - 1 = 0$ has roots α, β and γ.

(i) Write down the values of $\alpha\beta\gamma$ and $\alpha\beta + \alpha\gamma + \beta\gamma$. **1**

(ii) Evaluate $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **1**

(e) Evaluate $\displaystyle\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$ by using the substitution $u = x^2 + 1$. **3**

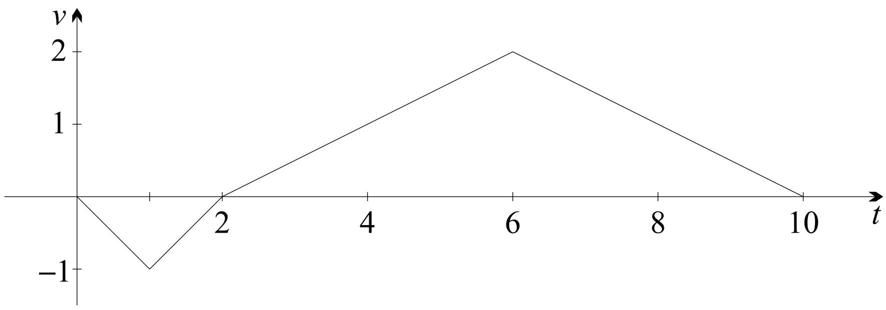
(f) Let $\dfrac{3x^3 - x^2 + 2x + 1}{x^2 - 1} = P(x) + \dfrac{R(x)}{x^2 - 1}$. **3**

Determine the polynomials $P(x)$ and $R(x)$ by long division.

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(x)$ has degree 3 and the zeroes are -1, 1 and 2. The graph of $y = P(x)$ passes through the point $(3, 16)$. Find $P(x)$ in factored form. **2**

(b)



The graph above shows the velocity–time graph of an object initially at the origin.

(i) When is the object stationary? **1**

(ii) During what period is the acceleration positive? **1**

(iii) At what time does the object return to the origin? **1**

(iv) At what time is it furthest from the origin? **1**

Exam continues next page ...

QUESTION NINE (Continued)

(c) A tennis player serves a ball from a height of 1·8 m. The ball initially travels horizontally with a velocity of 35 m/s. Neglect air resistance and assume that the acceleration due to gravity is 10 m/s^2. Let x metres be the horizontal distance the ball has travelled and let y metres be its height at times t seconds.

(i) Show by integration that the equations of motion of the ball are **2**

$$x = 35t \qquad \text{and} \qquad y = 1{\cdot}8 - 5t^2\,.$$

(ii) Find how long it takes for the ball to hit the ground. **1**

(iii) What is the horizontal distance travelled in that time? **1**

(iv) By how much does the ball clear the net which is 0·95 m high and 14 m from the player? (Ignore the dimensions of the ball.) **2**

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(x)$ is monic and has degree 3. It has a quadratic factor $(x^2 - 1)$. When $P(x)$ is divided by $(x - 2)$ the remainder is -9. **3**

Find $P(x)$ and hence solve the equation $P(x) = 0$.

(b) A car accelerates away from the origin so that its position x metres at time t seconds is given by

$$x = 20(t + e^{-0{\cdot}25t})\,.$$

(i) Find the velocity v and acceleration $\ddot{x}$ as functions of t. **2**

(ii) What is the eventual speed of the car? **1**

(iii) Sketch the velocity–time graph. **1**

(c) A particle is moving about the origin according to the rule

$$x = 4\sin(3t + \tfrac{\pi}{4})\,,$$

where x is the displacement in centimetres from the origin at time t seconds.

(i) Show that this is simple harmonic motion by showing that $\ddot{x} = -n^2 x$ for some value of n, and state the value of n. **2**

(ii) Write down the amplitude and period of the motion. **1**

(iii) Determine when the particle is at $x = 2$ for the first time. **2**

Exam continues overleaf ...

QUESTION ELEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) The roots of the equation $3x^3 - 19x^2 + 38x - 24 = 0$ form a geometric sequence. Solve the equation. **4**

(b) The equations of motion of a projectile fired at speed V and angle of elevation θ are:

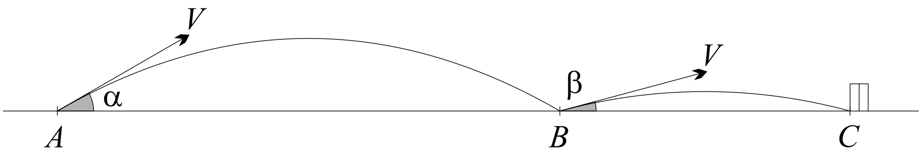
$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \tfrac{1}{2}gt^2\,,$$

where x is the horizontal distance in metres, y is the height in metres, and t is the time in seconds.

(i) Find the time of flight. **1**

(ii) Hence show that the range is $\dfrac{V^2\sin 2\theta}{g}$ metres. **1**

(c)



In a game of cricket, a fielder at A on the boundary throws a cricket ball at speed V and angle of elevation α to another fielder at B. The fielder at B instantly relays the throw at the same speed, but at an angle of elevation β, to a wicket-keeper at C. The three points A, B and C are collinear. The situation is shown in the diagram above.

Use your answers to part (b) to help answer the following.

(i) Write down an expression for the total time taken for the ball to arrive at C. **1**

(ii) Write down an expression for the total distance AC. **1**

(iii) The fielder at A throws another cricket ball at the same speed V and at an angle of elevation γ directly to the wicket-keeper at C. In the following you may assume that all three angles α, β and γ are less than 45° and greater than 0°.

(α) Explain why $\gamma > \alpha$ and $\gamma > \beta$. **1**

(β) Show that $\sin 2\gamma = \sin 2\alpha + \sin 2\beta$. **1**

(γ) Show that it takes longer to throw the ball directly to the wicket-keeper than to relay it via B. **2**

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \ \ n \neq -1; \ \ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \ \ x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \ \ a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \ \ a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \ \ a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \ \ a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \ \ a > 0, \ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

$$\text{NOTE}: \ \ln x = \log_e x, \ x > 0$$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

CANDIDATE NUMBER: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Multiple Choice (with possible source of errors)

Q 1 (D) (A) $P(-1) = 2$, (B) $P(-1) = 2$ sign error, (C) $k - 1 = 2$, then $k = 1$

Q 2 (A) (B) $\dfrac{d}{dx}\, v^2$, (C) $\displaystyle\int \tfrac{1}{2} v^2\, dx$, (D) $\displaystyle\int v^2\, dx$

Q 3 (D) (A) $x(0) = 0$, $n = T$, (B) $n = T$, (D) $x(0) = 0$

Q 4 (C) (A) $\frac{b}{a}$, $-\frac{d}{a}$, (B) $\frac{b}{a}$, $\frac{d}{a}$, (D) $-\frac{b}{a}$, $\frac{d}{a}$

Q 5 (B) (A) wrong quadrant, (C) reciprocal, wrong quadrant, (D) reciprocal

Q 6 (A) (B) amplitude, (C) misread: $3 - 4\sin t$, (D) add amplitude to 3

Q 7 (C) (A) $\dot{y}$, (B) $\dot{y}(0)$, (D) initial speed

QUESTION EIGHT (12 marks)

(a)
$$\begin{aligned} P(-1) &= -6 + 5 + 2 - 1 \\ &= 0 \qquad \text{so } (x+1) \text{ is a factor} \end{aligned}$$
✓

(b)
$$-1 \le 2x \le 1$$
so $-\frac{1}{2} \le x \le \frac{1}{2}$ ✓

(c)
$$y = \tan^{-1} 3x$$
so $y' = \dfrac{3}{1 + 9x^2}$ ✓✓

[(1) for derivative of $\tan^{-1}$ and (1) for chain rule.]

(d) (i)
$$\alpha\beta\gamma = 1$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = 0$$
✓

(ii)
$$\begin{aligned} \tfrac{1}{\alpha} + \tfrac{1}{\beta} + \tfrac{1}{\gamma} &= \tfrac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} \\ &= 0 \end{aligned}$$
✓

(e) $$I = \int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$$

Let $u = x^2 + 1$

at $x = 0, \quad u = 1$

$x = \sqrt{3} \quad u = 4$

and $\frac{1}{2}du = x\,dx\,,$ ✓

so $$I = \int_1^4 \tfrac{1}{2}\sqrt{u}\,du$$ ✓

$$= \tfrac{1}{3}\left[u^{3/2}\right]_1^4$$

$$= \tfrac{7}{3}$$ ✓

(f)

$$\begin{array}{r|l} & \quad\quad\quad\quad\ 3x - 1 \\ x^2 - 1 & 3x^3 - x^2 + 2x + 1 \\ & \underline{3x^3 \quad\quad - 3x} \\ & \quad\ -x^2 + 5x + 1 \\ & \quad\ \underline{-x^2 \quad\quad + 1} \\ & \quad\quad\quad\quad 5x \end{array}$$

✓ ✓

so $P(x) = 3x - 1$ and $R(x) = 5x$ ✓

Total for Question 8: 12 Marks

QUESTION NINE (12 marks)

(a) Let $P(x) = a(x+1)(x-1)(x-2)$ ✓

$(3, 16)$ is on $y = P(x)$ so

$$16 = a \times 4 \times 2 \times 1$$

thus $a = 2$

and $P(x) = 2(x+1)(x-1)(x-2)$ ✓

(b) (i) $t = 0, 2, 10$ ✓

(ii) $1 < t < 2$ or $2 < t < 6$ ✓

[Accept $1 < t < 6$ or similar.]

(iii) By equal areas, $t = 4$ ✓

(iv) $t = 10$ ✓

(c) (i)

$\ddot{x} = 0$
so $\dot{x} = C_1$
At $t = 0$, $35 = C_1$
so $\dot{x} = 35$
thus $x = 35t + C_2$
At $t = 0$, $0 = 0 + C_2$
so $x = 35t$ ✓

$\ddot{y} = -10$
so $\dot{y} = C_3 - 10t$
At $t = 0$, $0 = C_3 - 0$
so $\dot{y} = -10t$
thus $y = C_4 - 5t^2$
At $t = 0$, $1{\cdot}8 = C_2 - 0$
so $y = 1{\cdot}8 - 5t^2$ ✓

(ii) At $y = 0$ $\quad 5t^2 = \frac{9}{5}$
so $t = \frac{3}{5}$ $\quad (t > 0)$ ✓

(iii) At $t = \frac{3}{5}$ $\quad x = 21$ ✓

(iv) At $x = 14$ $\quad 35t = 14$
so $t = \frac{2}{5}$ ✓
Now $y(\frac{2}{5}) = \frac{9}{5} - 5 \times (\frac{2}{5})^2$
$= 1$
Thus the ball clears the net by $0{\cdot}05\,\text{m}$ ✓

Total for Question 9: 12 Marks

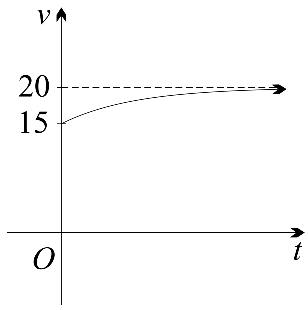
QUESTION TEN (12 marks)

(a) Let $P(x) = (x^2 - 1)(x - \alpha)$ ✓
Now $P(2) = -9$
so $3(2 - \alpha) = -3$
thus $\alpha = 5$ ✓
hence $P(x) = (x + 1)(x - 1)(x - 5)$.
Thus $P(x) = 0$ has solutions $x = -1, 1, 5$. ✓

(b) (i) $v = 20(1 - \frac{1}{4}e^{-0{\cdot}25t})$
$= 5(4 - e^{-0{\cdot}25t})\,\text{m}\,\text{s}^{-1}$ ✓
$\ddot{x} = \frac{5}{4}e^{-0{\cdot}25t}\,\text{m}\,\text{s}^{-2}$ ✓

(ii) $\lim_{t\to\infty} v = \lim_{t\to\infty} 5(4 - e^{-t/4})$
$= 5(4 - 0)$
$= 20\,\text{m}\,\text{s}^{-1}$ ✓

(iii) $\boxed{\surd}$



(c) (i) $x = 4\sin(3t + \frac{\pi}{4})$

$\dot{x} = 12\cos(3t + \frac{\pi}{4})$

$\ddot{x} = -36\sin(3t + \frac{\pi}{4})$

$= -9x$ $\boxed{\surd}$

with $n = 3$ $\boxed{\surd}$

(ii) amplitude $= 4$

period $= \frac{2\pi}{3}$ $\boxed{\surd}$

(iii) At $x = 2$, $\sin(3t + \frac{\pi}{4}) = \frac{1}{2}$

so $3t + \frac{\pi}{4} = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$ $\boxed{\surd}$

thus $t = -\frac{\pi}{36}, \frac{7\pi}{36}, \ldots$

Hence the first positive solution is $t = \frac{7\pi}{36}$ $\boxed{\surd}$

Total for Question 10: 12 Marks

QUESTION ELEVEN (12 marks)

(a) Let the roots be $\frac{a}{r}$, a and ar, then:

$\frac{a}{r} \times a \times ar = \frac{24}{3}$ (product of roots)

so $a = 2$ $\boxed{\surd}$

and $\frac{a}{r} + a + ar = \frac{19}{3}$ (sum of roots)

or $6r^2 - 13 + 6 = 0$ $\boxed{\surd}$

thus $(3r - 2)(2r - 3) = 0$

so $r = \frac{2}{3}$ or $\frac{3}{2}$ $\boxed{\surd}$

In both cases the roots are $\frac{4}{3}, 2, 3$ $\boxed{\surd}$

(b) (i) At $y = 0$ $t(V\sin\theta - \frac{1}{2}gt) = 0$

so $t = \frac{2V\sin\theta}{g}$ $(t \neq 0)$ $\boxed{\surd}$

(ii) From (i) $x = V \times \frac{2V\sin\theta}{g} \times \cos\theta$

$= \dfrac{V^2 \times 2\sin\theta\cos\theta}{g}$ ✓

$= \dfrac{V^2\sin 2\theta}{g}$.

(c) (i) Time $AC = \frac{2V\sin\alpha}{g} + \frac{2V\sin\beta}{g}$

$= \frac{2V}{g}(\sin\alpha + \sin\beta)$ ✓

(ii) Range $AC = \dfrac{V^2\sin 2\alpha}{g} + \dfrac{V^2\sin 2\beta}{g}$

$= \frac{V^2}{g}(\sin 2\alpha + \sin 2\beta)$ ✓

(iii) (α) Since range $AC >$ range AB

$\sin 2\gamma > \sin 2\alpha$ ✓

so $\gamma > \alpha$ (since 2γ and 2α are acute)

and likewise, $\gamma > \beta$.

[Accept any other valid argument.]

(β) From the range of AC

$\frac{V^2}{g}\sin 2\gamma = \frac{V^2}{g}(\sin 2\alpha + \sin 2\beta)$ ✓

or $\sin 2\gamma = \sin 2\alpha + \sin 2\beta$

(γ) $\sin 2\gamma = \sin 2\alpha + \sin 2\beta$

so $2\sin\gamma\cos\gamma = 2\sin\alpha\cos\alpha + 2\sin\beta\cos\beta$ ✓

Now $\gamma > \alpha$ so $\cos\gamma < \cos\alpha$ (cos is decreasing for acute angles)

and $\gamma > \beta$ so $\cos\gamma < \cos\beta$ (again, cos is decreasing for acute angles)

thus $2\sin\gamma\cos\gamma > 2\sin\alpha\cos\gamma + 2\sin\beta\cos\gamma$

hence $\sin\gamma > \sin\alpha + \sin\beta$ ✓

thus $\frac{2V}{g}\sin\alpha > \frac{2V}{g}(\sin\alpha + \sin\beta)$

That is, the direct throw takes longer than the relayed throw.

Total for Question 11: 12 Marks

Out of interest, here is another solution to the last part.

Let $AB = a$ and $BC = b$ then the time for the relayed throw is

$$\begin{aligned} t_1 &= \frac{a}{V\cos\alpha} + \frac{b}{V\cos\beta} \\ &= \frac{a\cos\beta + b\cos\alpha}{V\cos\alpha\cos\beta} \end{aligned}$$

and the time for the direct throw is

$$t_2 = \frac{a+b}{V\cos\gamma}\,.$$

Now
$$\begin{aligned} t_2 - t_1 &= \frac{a+b}{V\cos\gamma} - \frac{a\cos\beta + b\cos\alpha}{V\cos\alpha\cos\beta} \\ &= \frac{(a+b)\cos\alpha\cos\beta - a\cos\beta\cos\gamma - b\cos\alpha\cos\gamma}{V\cos\alpha\cos\beta\cos\gamma} \\ &= \frac{a\cos\beta(\cos\alpha - \cos\gamma) + b\cos\alpha(\cos\beta - \cos\gamma)}{V\cos\alpha\cos\beta\cos\gamma}\,. \end{aligned}$$

But $\gamma > \alpha$ and $\gamma > \beta$

so $\cos\alpha - \cos\gamma > 0$ and $\cos\beta - \cos\gamma > 0$

Hence $t_2 - t_1 > 0\,,$

that is, the direct throw is slower than the relayed throw.

BR/DNW